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AMTH140 **Lecture 14**

Eulerian Paths and Circuits

March 28, 2006

Reading: Lecture Notes §9.3; Epp §11.2

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Eulerian Paths and Circuits

An **Eulerian circuit** in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once.

A graph which has an Eulerian circuit is called an **Eulerian graph**.

An **Eulerian path** in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once.

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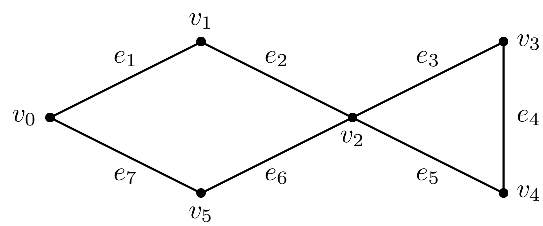

The walk

$$v_0 e_1 v_1 e_2 v_2 e_3 v_3 e_4 v_4 e_5 v_2 e_6 v_5 e_7 v_0$$

is an Eulerian circuit.

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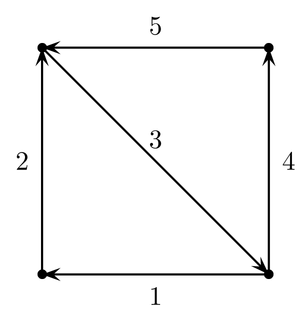

An Eulerian path which is not a circuit.

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Euler's Theorem

1. *A connected graph G is Eulerian, i.e. it has an Eulerian circuit, if and only if every vertex has even degree.*

2. *A connected graph G has an Eulerian path, but not an Eulerian circuit, if and only if G has exactly two vertices of odd degree. The Eulerian path must start at one of the two odd degree vertices and finish at the other.*

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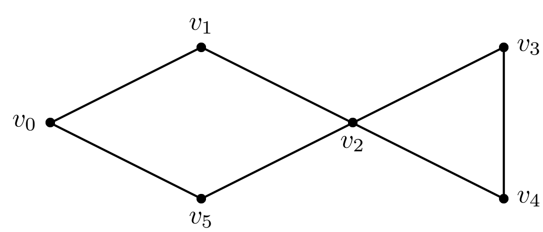
Examples



All the vertices have even degree, (either 2 or 4) therefore the graph has an Eulerian circuit.

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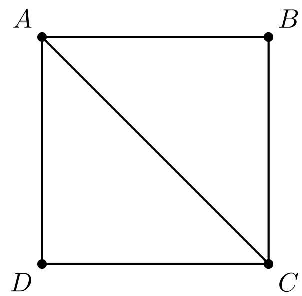

Vertices A and C have degree 3, while vertices B and D have degree 2. Since all but two vertices have even degree, by Euler's theorem there is an Eulerian path. It must start at one of A or C and end at the other.

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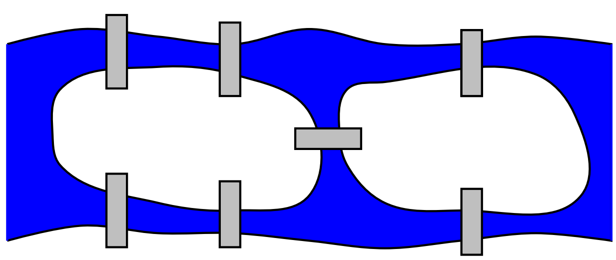
Example: The Königsberg Bridge Problem



Is it possible to walk round the town crossing each of the seven bridges exactly once?

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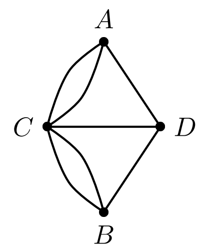
Let A and B denote the two banks of the river, and C and D the two islands. These can be considered vertices of a graph with the bridges as edges.



The problem of finding a walk which crosses each of the bridges exactly once, is that of finding an Eulerian circuit or path.

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All four vertices have odd degree

$\deg(A) = 3$, $\deg(B) = 3$, $\deg(C) = 5$, $\deg(D) = 3$.

Therefore, by Euler's theorem, there is no Eulerian circuit or path.

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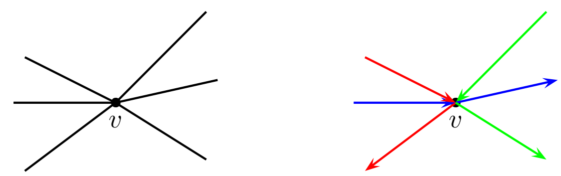
Proof of Euler's Theorem

Assume G is a graph with an Eulerian circuit, we will show that each vertex has even degree.

Choose any vertex v, and consider the edges entering and leaving v as we perform our walk:



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For each edge that enters v there must be another edge which leaves v. Since each edge is traversed once, there must be an even number of edges incident on v.

If we consider paths rather than circuits, then exactly two vertices, the starting and ending vertices, will have odd degree.

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Now we should prove the other direction of the theorem:

If every vertex of a connected graph G has even degree then G has an Eulerian circuit.

If all but two vertices of a connected graph G have degree then G has an Eulerian path.

Rather than giving a proof, we will give an algorithm, called Fleury's algorithm, for constructing an Eulerian path or circuit.

The proof of Euler's theorem in Epp's book (pp 672-673) can be used to justify Fleury's algorithm. There is a different proof, using mathematical induction, in the Lecture Notes.

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Fleury's Algorithm

Here is the algorithm for an Eulerian path:

1. Choose one of the two vertices of odd degree as the starting point.
2. Travel over any edge whose removal will not break the graph into disconnected components.
3. Colour the edge just traversed, and then travel over any edge whose removal will not break the remaining sub-graph into disconnected components.
4. Repeat until all edges are coloured (i.e. traversed).

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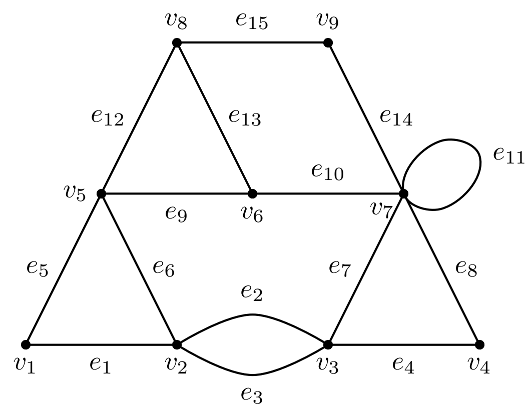

There are two vertices of odd degree, v_6 and v_8, all others have even degree.

Choose v_8 as the starting vertex.

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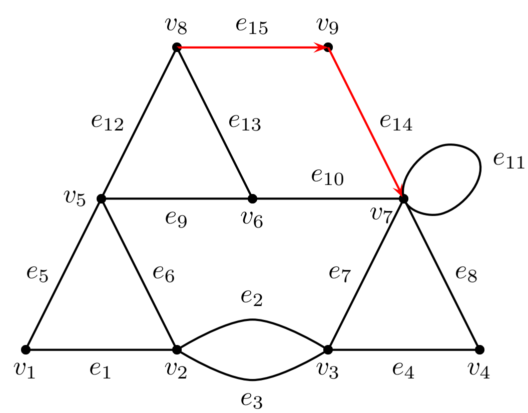
We can move along e_{12}, e_{13}, or e_{15}. Choose e_{15}, then our next move must be along e_{14}.



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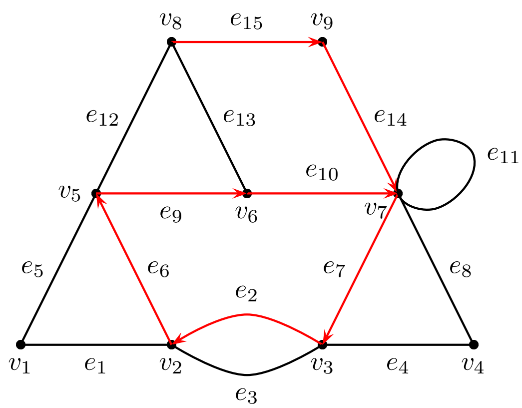
Now move along e_7, e_2, e_6, e_9 and e_{10} – there are many other choices.



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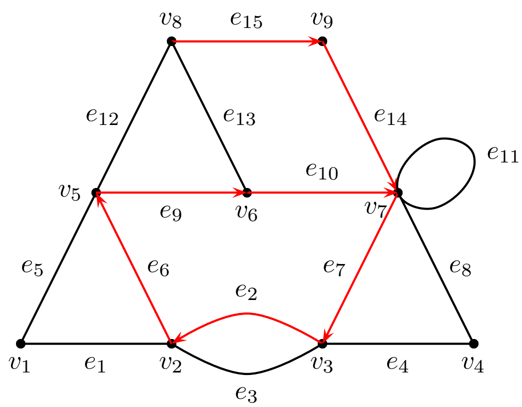
We now have two choices, move along the loop e_{11}, or the edge e_8. If we moved along e_8, it would disconnect the loop e_{11} from the rest of the graph



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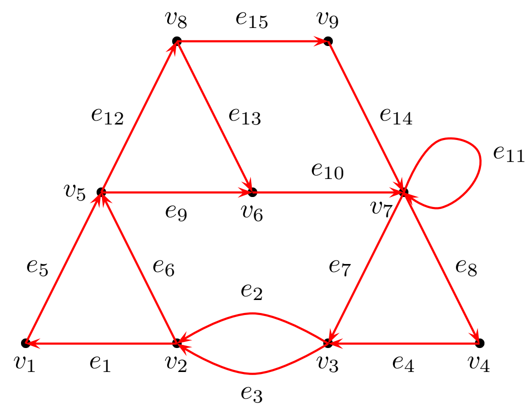
Therefore we must move along the loop e_{11} first. After that our remaining move are determined for us:



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One interesting thing to note about Fleury's algorithm is that at each step, if remove the edges already used, we still have a graph which satisfies the conditions of Euler's theorem, i.e. exactly two vertices of odd degree.

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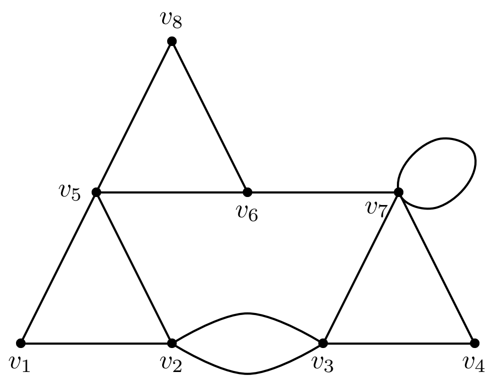
At this stage we are at the vertex v_7 and vertices v_6 and v_7 have odd degree.



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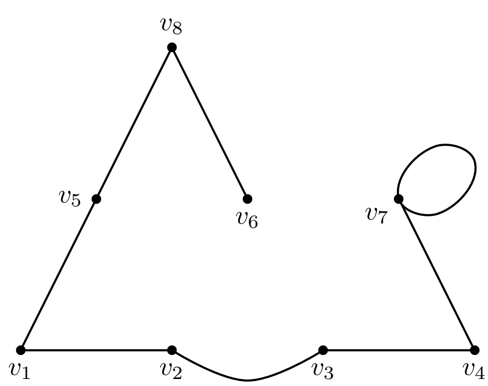
At this stage we again are at the vertex v_7 and again it is vertices v_6 and v_7 which have odd degree.



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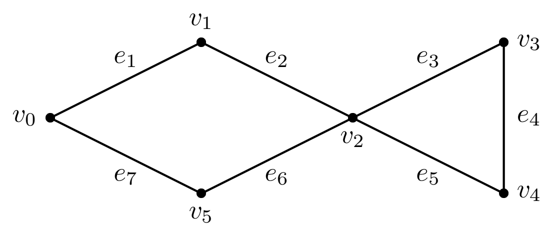
Finding an Eulerian Circuit

Fleury's algorithm also finds Eulerian circuits, just start at any vertex and use the algorithm for paths:



We will start at v_0.

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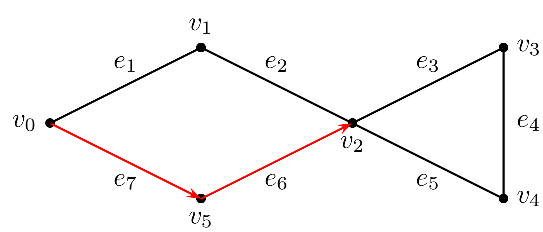
Our first move is either e_1 or e_7. Pick e_7, then our next move must be e_6.



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Our next move is either e_2, e_3 or e_5. We cannot take e_2 as this will disconnect the graph. Choose e_5 and not that after that all our moves are determined.