Recurrence

S1. Write ← last week

S2. SOLVE ← this week

CLASSIFICATION

$f(n) = a_n + h_1(n)\, a_{n-1} + \cdots + h_k(n)\, a_{n-k}$: Recurrence Relatⁿ

KNOWN $a_m, a_{m+1}, \ldots, a_{m+k-1}$: INITIAL CONDITIONS

$0 = F_n - F_{n-1} - F_{n-2}$ k: depth of RR

↳ $f(n)=0$ $F_0=0, F_1=1$ $k=2$

$h_i(\cdot)=1$

Const. coeff. RR

If $f(n)=0$: Homegeneous RR

If $h_i(n) = \text{const.}\ \forall i=1,\ldots,k$: Constant Coef. R.R.

In general $h_i(\cdot)$'s are functions

EG 2

$n! = n \cdot (n-1)!$ → next page

$0 = n! - n\cdot(n-1)! \longleftrightarrow f(n) = a_n + h_1(n)\, a_{n-1}$

↳ Homegenous Nonconstant RR

with depth 1 $\longleftrightarrow$ $a_1 = 1! = 1$ OR $a_0 = 0! = 1$

INITIAL COND.

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

$a_0 = 2,\ a_1 = 5$ $\longrightarrow$ Homegeneous Const. Coeff. RR with depth $= 2$

Solve) $a_n = 2^n + 3^n$

M1. Using generating functions

M2. Using characteristic equations

$a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad \forall n = 2, 3, 4, \dots$ (✯)

$a_0 = 2, \; a_1 = 5$

Claim 1: $a_n = 2^n$

Claim 2: $a_n = 3^n$

$3^n - 5[3^{n-1}] + 6[3^{n-2}] \overset{?}{=} 0$

Divide by 3^{n-2}

$9 - 5(3) + 6 = 0$ ✓

(✯): $2^n - 5(2^{n-1}) + 6(2^{n-2}) \overset{?}{=} 0$

Dividing (✯) by 2^{n-2}

$4 - 5(2) + 6 = 0$ ✓

Superposition Principle:

If $a_n = s(n)$ and $a_n = t(n)$ are two solutions to RR (✯), then the general solution is

$a_n = c_1 s(n) + c_2 t(n)$

where c_1 & c_2 are constants determined from initial cond^n.

In our example problem

$a_0 = 2,\ a_1 = 5$ $\qquad a_n = c_1 2^n + c_2 3^n$ (**)

$\hookrightarrow n=0 \qquad a_0 = c_1 2^0 + c_2 3^0 = \boxed{c_1 + c_2 = 2}$ (1)

$\hookrightarrow n=1 \qquad a_1 = c_1 2^1 + c_2 3^1 = \boxed{2c_1 + 3c_2 = 5}$ (2)

$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Longrightarrow c_1 = 1 = c_2$ $\qquad \therefore \quad a_n = 2^n + 3^n$

M1: Using g(x)

$$a_n x^n - 5a_{n-1}x^n + 6a_{n-2}x^n = 0x^n = 0 \quad \forall n = 2, 3, 4, \dots$$

sum all the equations to get only one eq$^{\underline{n}}$

$$\sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1}x^n + 6\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\sum_{k=2}^{\infty} a_k x^k - 5x\sum_{k=1}^{\infty} a_k x^k + 6x^2 \underbrace{\boxed{\sum_{k=0}^{\infty} a_k x^k}}_{g(x)} = 0$$

$$g(x) - a_0x^0 - a_1x^1 - 5x\left[g(x) - a_0x^0\right] + 6x^2\left[g(x)\right] = 0$$

$$\left.\begin{matrix} a_0 = 2 \\ a_1 = 5 \end{matrix}\right\} \Rightarrow g(x) - 2 - 5x - 5x\left[g(x) - 2\right] + 6x^2 g(x) = 0$$

$$g(x)\left[1 - 5x + 6x^2\right] = 2 + 5x - 10x$$

$$\Rightarrow g(x) = \frac{2 - 5x}{(1-2x)(1-3x)}$$

$$g(x) = \frac{2-5x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x} \quad (\star\star\star)$$

Coef x^0: $A+B=2$ (1)

 x^1: $-3A-2B=-5$ (2)

$$\left.\begin{matrix} A+B=2 & (1) \\ -3A-2B=-5 & (2) \end{matrix}\right\} A=1=B$$

$$\frac{1}{1-y} \left(\text{From Formula Sheet} \rightarrow\right) = 1+y+y^2+y^3+\dots$$

$$g(x) = \frac{1}{1-2x} + \frac{1}{1-3x} \text{ from } \star\star\star \longrightarrow a_n = 2^n + 3^n$$

$$g(x) = \left(1+2x+4x^2+\dots+2^n x^n+\dots\right) + \left(1+3x+9x^2+\dots+3^n x^n+\dots\right)$$

$$g(x) = (1+1)x^0 + (2+3)x^1 + (2^2+3^2)x^2 + \dots + \underbrace{(2^n+3^n)}_{a_n}x^n + \dots$$

Method 2: Characteristic Equation

$$f(n) = a_n + h_1(n)\, a_{n-1} + \cdots + h_k(n)\, a_{n-k}$$

$$\alpha^k + h_1(n)\, \alpha^{k-1} + \cdots + h_k(n)\, \alpha^0 = 0$$

The characteristic Eq$^{\underline{n}}$ is a polynomial of order k

depth

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad (*)$$

$$\alpha^2 - 5\alpha^1 + 6\alpha^0 = 0 \quad (CE)$$

$$\alpha^2 - 5\alpha + 6 = 0$$

Find the roots α_1, α_2 of the CE

$$(\alpha - 3)(\alpha - 2) = 0$$

$$\alpha_1 = 2 \;\&\; \alpha_2 = 3$$

If all the roots $\alpha_1, \ldots, \alpha_k$ of CE are distinct

$$a_n = c_1 \alpha_1^n + c_2 \alpha_2^n + \cdots + c_k \alpha_k^n$$

such that $c_1, \ldots, c_k$ are determined from initial conditions

$a_n - 5a_{n-1} + 6a_{n-2} = 0 \Rightarrow \alpha_1 = 2, \alpha_2 = 3$

$a_0 = 2, \; a_1 = 5$

$\Rightarrow a_n = c_1 2^n + c_2 3^n$

$\Rightarrow c_1 = 1 = c_2$

$$a_n = 2^n + 3^n$$

1) If the roots of CE is not distinct:

$$CE: \sum_{i=1}^{\ell} (\alpha - \alpha_i)^{m_i} = 0 \quad \exists\ 1, 2, \ldots, \ell \text{ different roots } \alpha_1, \ldots, \alpha_\ell$$

Root α_i is repeated m_i many times

General solution

$$\to a_n = \cdots + \overbrace{c_{1,8}\,\alpha_i^n + c_{1,9}\, n\,\alpha_i^{n-1} + \cdots + c_{2,8}\, n^{r-1} \alpha_i^{n-r}}^{m_i \text{ terms}} + \ldots$$

2) If RR is inhomogeneous

Then $a_n = b_n + h_n$

b_n: general sol^n of the reccurrence part

h_n: sol^n of the homogeneous part

$a_n + a_{n-2} = 0 \quad \forall n = 2, 3, \dots$ Solve using CE

$a_1 = 0$

$a_2 = 1$

$a_3 = 0, \; a_4 = -1, \; a_5 = 0$

S1. Write CE: $\alpha^2 + \alpha^0 = 0$

$$\boxed{\alpha^2 + 1 = 0}$$

S2. Find the roots $\alpha = \pm\sqrt{-1} \Rightarrow \alpha_1 = +i$

where $i^2 = -1$ $\qquad \alpha_2 = -i$

S3. (If the roots are distinct) Write the general sol$\underline{n}$

$$\boxed{a_n = c_1 (+i)^n + c_2 (-i)^n} = -\frac{1}{2}\left[(+i)^n + (-i)^n\right]$$

S4. Find constants using initial cond$\underline{n}$s

$a_1 = 0 \Rightarrow c_1 (\cancel{i}) - c_2 (\cancel{i}) = 0 \Rightarrow c_1 = c_2$

$a_2 = 1 \Rightarrow -c_1 - c_1 = 1 \Rightarrow c_1 = \frac{1}{2}$

Fibonacci: $f_n - f_{n-1} - f_{n-2} = 0 \quad n = 2, \ldots,$

C.E. $\quad \alpha^2 - \alpha - 1 = 0 \qquad \alpha_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$\Delta = b^2 - 4ac$ $\qquad$ roots of $ax^2 + bx + c = 0$

$= 1 + 4 = 5$

$\alpha_1 = \dfrac{+1 + \sqrt{5}}{2}, \quad \alpha_2 = \dfrac{1 - \sqrt{5}}{2}$

$\rightarrow$ continue (as an exercise)

$f_n = c_1 \left[\dfrac{1+\sqrt{5}}{2}\right] + c_2 \left[\dfrac{1-\sqrt{5}}{2}\right]$

$\rightarrow \ldots$