S1. Write $g(x)$ $\longleftarrow$ Last Week

S2. Extract coeff. $\longleftarrow$ THIS WEEK

EXAMPLE 1 Find the coefficient of x^{20} in $(x^3+x^4+\dots)^3$

$$g(x) = (x^3+x^4+x^5+\dots)^3$$
$$= (x^3[1+x+x^2+\dots])^3 = x^9 \underbrace{[1+x+x^2+\dots]^3}$$

We are interested in the coefficient of x^{11} in $h(x)$

$$h(x) = (1+x+x^2+\dots)^3$$

From the formula sheet

$$h(x) = \binom{3-1+0}{0}x^0 + \binom{3-1+1}{1}x^1 + \dots + \binom{3-1+k}{k}x^k + \dots$$

$$\binom{3-1+11}{11} = \binom{13}{11} = \frac{13 \cdot 12 \cdot \cancel{11!}}{\cancel{11!}\,2!} = 13(6) = 78$$

Example 2 $\quad x^9$ in $(1+x+\cdots+x^5)^4 = g(x)$

$$g(x) = \underbrace{[1-x^6]^4}_{l(x)} \cdot \underbrace{(1+x+x^2+\cdots)^4}_{k(x)}$$

$\boxed{9^{th}}$ convolution of $l(x) \cdot k(x)$

$$l(x) = (1-x^6)^4 = \binom{4}{0} - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + \binom{4}{4}x^{24}$$

$$k(x) = \binom{4-1+0}{0}x^0 + \binom{4-1+1}{1}x^1 + \cdots + \binom{4-1+k}{k}x^k + \cdots$$

$$l_0 k_9 + l_1 k_8 + \cdots + l_8 k_1 + l_9 k_0 \qquad x^9$$

$$= l_0 k_9 + l_6 k_3 = \binom{4}{0}\binom{12}{9} - \binom{4}{1}\binom{6}{3}$$

$$= \frac{12(11)10}{3 \cdot 2 \cdot 1} - 4\,\frac{6(5)4}{3 \cdot 2}$$

$$= 220 - 80 = 140$$

Formula Sheet

$$(1+x+x^2+\cdots) = \frac{1}{1-x}$$

$$(1+x+x^2+\cdots+x^k) = \frac{1-x^{k+1}}{1-x} = [1-x^{k+1}][1+x+x^2+\cdots]$$

$$(1-y^k)^n = \binom{n}{0} - \binom{n}{1}y^k + \binom{n}{2}y^{2k} + \cdots + \overset{(-1)^r}{\binom{n}{r}}\, y^{rk} + \cdots + \overset{(-1)^n}{\binom{n}{n}}\, y^{nk}$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^1$$

$$(1+x+\cdots)^n = \binom{n-1+0}{0} + \binom{n-1+1}{1}x + \cdots + \binom{n-1+k}{k}x^k + \cdots$$

$$(x_1+\cdots+x_\ell)^n = \sum_{\substack{k_1,\ldots,k_\ell \\ k_1+\cdots+k_\ell=n}} \binom{n}{k_1,\ldots,k_\ell} x_1^{k_1}\cdots x_\ell^{k_\ell}$$

RECURRENCE

King Shirnam of India wants to → grand visors of Ben Dahir for inventing CHESS

made a *modest* request

"I would like to have one grain (seed) of wheat in the first cell on the chess board. For the next cell, I want just the double"

$t_1 = 1$

King ordered as LET it BE!

t_i = total # of seeds in i^{th} cell

$t_1 = 1$, $t_2 = 2t_1 = 2$, $t_3 = 2 \cdot t_2 = 4$, $t_4 = 2t_3 = 8$, $t_5 = 16$, $t_6 = 32$

$t_n = 2t_{n-1} = 2^2 t_{n-2} = 2^{n-1} t_1 = 2^{n-1}$

$t_{64} = 2^{63} = 18 \times 10^{18}$

Compound Interest

i = annual interest rate

$$A_1 = (1+i)A_0$$

A_0 = investment at the begining

$$A_2 = (1+i)A_1 = (1+i)^2 A_0$$

$\vdots$

$$A_k = (1+i)^k A_0$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$: A two dimensional Recursion

$$n! = f(n) = n\, f(n-1) = n \cdot (n-1)! = n\left[(n-1)(n-2)!\right]$$

$$= \cdots = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

This week: Define recursions

Next week: Solve recursions

Codewords from alphabet {0,1,2,3} are recognized LEGITIMATE iff $\exists$ even # of 0s

a_k = # of legitimate codewords of length k [string "$\underline{k}$ $\underline{k-1}$ ... $\underline{1}$"]

$a_k \overset{?}{=} f(a_{k-1})$

a_{k-1} : # legitimate of length $(n-1)$

"$\underline{4}$ $\underline{4}$ ... $\underline{4}$"

$\xleftarrow[\text{of prod}]{\text{Rule}}$

$4^{k-1} - a_{k-1}$: # of illegal codewords

4^{k-1} : # of different codewords of size $(n-1)$

$$a_k = 1 \cdot \left[4^{k-1} - a_{k-1}\right] + 3\, a_{k-1}$$

Transmit messages (dots, dashes) over a channel using two symb

dots = a, dashes = b

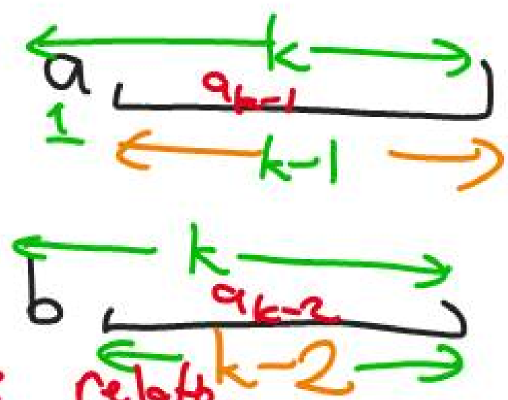
Signal a takes 1 t.u.

b takes 2 tu(s)

a_k = # of different messages that can be sent in k time units

If the last signal is dot

dash



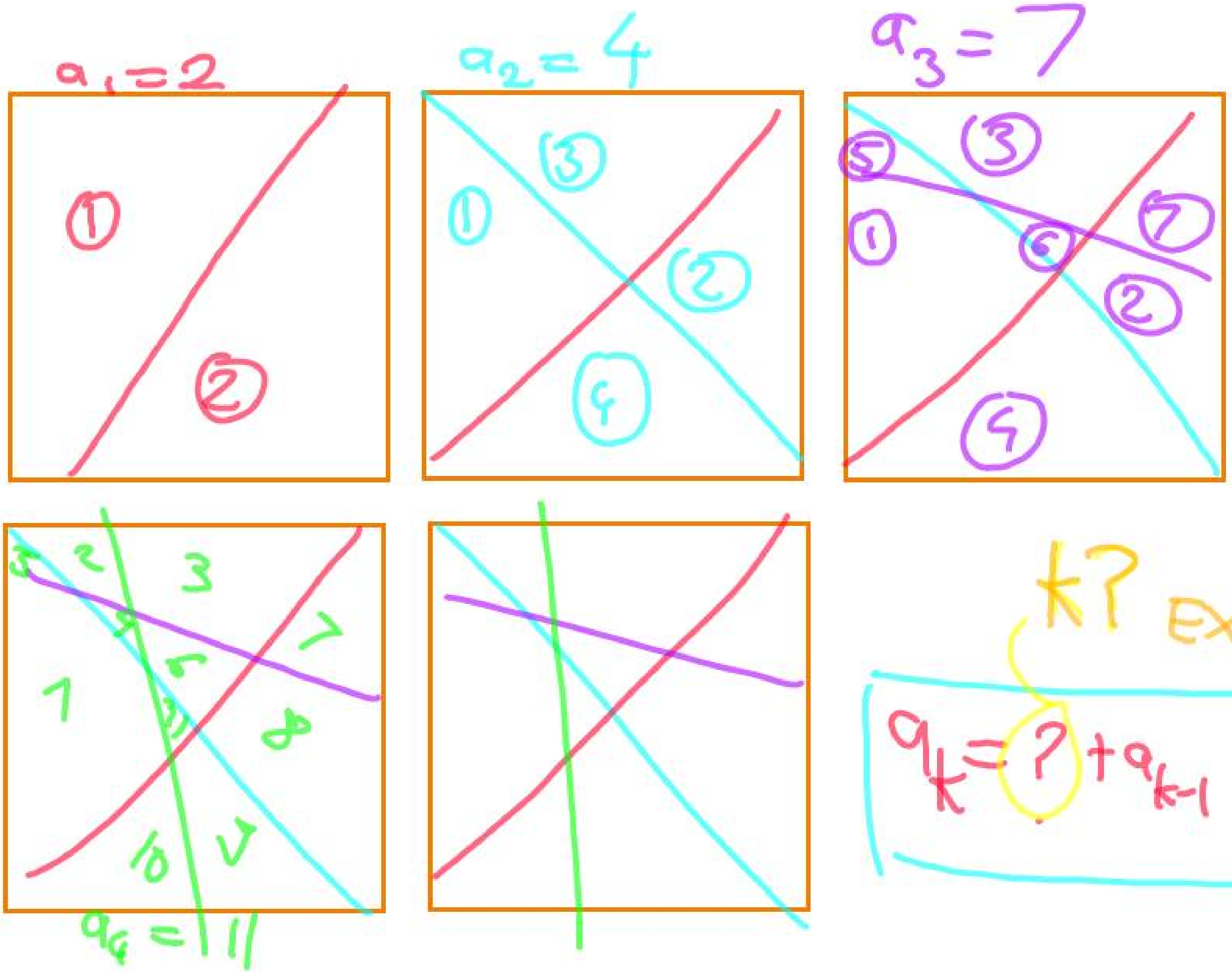
$a_k = a_{k-1} + a_{k-2}$: reccurrence relation

$a_0 = 0$ $\quad a_1 = 1$ only a dot

$a_2 = 2$ a dash OR two dots

$\Big\}$ Boundary Conditions

Regions in a plane if there k pairwise intersecting lines



k? EXERCISE

$$\boxed{a_k = ? + a_{k-1}}$$

1202 Leanardo di Pisa Liber Abaci

"Breeding of Rabbits"

- Rabbits never DIE.
- We start with 1 pair (opposite sex) of rabbits.
- Rabbits give birth in two months, one pair/pair
- New born pair become adults in a month and produce their own children in two months

# adult pair	# 1-month old pair	# baby pairs	Pairs TOTAL F_t	Month (t)
1	—	—	1	0
1	—	—	1	1
1	—	1	2	2
1	1	1	3	3
2	1	2	5	4
3	2	3	8	5

$F_0 = 1$ (0)
$F_1 = 1$
$F_2 = 2$
$F_3 = 3$
$F_4 = 5$
$F_6 = 8$
$F_7 = 13$
$F_8 = 21$
$F_9 = 34$
$\vdots$

$$F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

F_k / F_{k-1}	$3/2$	$5/3$	$8/5$	$13/8$	$21/13$	$34/21 \cdots$	GOLDEN RATIO
	1.5	$1.\bar{6}$	1.6	1.67	1.615	1.619	τ

FIBONACCI SEQUENCE

$$F_k = F_{k-1} + F_{k-2} \quad F_0 = 1,\ F_1 = 1$$