GENERATING FUNCTIONS

$g(x) = (x^1 + x^2)(x^2 + x^3) = x^3 + x^4 + x^4 + x^5 = 1x^3 + \boxed{2}\,x^4 + 1\,x^5$

$x_1 + x_2 = k \quad (\star)$

$1 \le x_1 \le 2$ INT $\Rightarrow x_1 = 1, 2$

$2 \le x_2 \le 3$ INT $\Rightarrow$ $x_2 = 2, 3$

How many ways are there to have a sum $(\star)$ of k?

$k = 3 \Rightarrow$ there is only one way

4 2 ways

5 One way

The generating function for a combinatorial problem

$x_1 + \cdots + x_n = k$ (★) is a power series $\left[\sum_{i=0}^{\infty} a_i x^i\right]$

$x_i \in S_i$

such that the coefficient of the term

x^k (namely a_k) is the answer

of the question:

How many different ways are there

to have a sum of k in (★)

$\boxed{\text{Hw 1 - Q2b}}$

$x_1 + \cdots + x_n = k$

$x_i = 0$ or 1 $\rightarrow$ $x^0 + x^1$

$$\underbrace{(x^0+x^1)}_{x_1}\underbrace{(x^0+x^1)}_{x_2}\cdots\underbrace{(x^0+x^1)}_{x_n} = g(x) = (1+x)^n$$

Pascal's Triangle

1 1
1 2 1
1 3 3 1
...

Binomial Theorem

$$(1+x)^n = \binom{n}{0}1 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \underbrace{\binom{n}{k}}_{a_k}x^k + \cdots + \binom{n}{n}x^n$$

$\binom{n}{k}$: k-combination

S1. Write down $g(x)$ ← Week 3

S2. Extracting coefficients: a_n ← Week 4

Power Series

$$g(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_k x^k + \dots$$

$$h(x) = \sum_{i=0}^{\infty} b_i x^i = b_0 + b_1 x + b_2 x^2 + \dots + b_k x^k + \dots$$

$$k(x) = g(x) + h(x) = (a_0 + b_0) + \dots + (a_k + b_k) x^k + \dots$$

$$\ell(x) = g(x) \cdot h(x) = \sum_{i=0}^{\infty} d_i x^i$$

$d_i = i^{th}$ convolution $= \sum_{j=0}^{i} a_j b_{i-j} = a_0 b_i + a_1 b_{i-1} + \dots + a_j b_{i-j} + \dots$

$a_i b_0$

$$(2x^4+5x^5-2x^6+3x^7)(4-2x+3x^2+4x^4+5x^6)$$

What is the coefficient of x^6?

$$2(5)-2(4)$$

A nice way of counting:

$$g(x) = 1 + \cancel{x} + \cancel{x^2} + \cdots + x^k + \cdots = ? \qquad \frac{1}{1-x}$$

$$xg(x) = \cancel{x} + \cancel{x^2} + x^3 + \cdots + x^{k+1} + \cdots$$

$$= 1$$

$$1 = g(x)[1-x]$$

S1. Writing $g(x)$

Example 1

Hw 2a) $x_1 + \cdots + x_n = k$

$x_i \geq 0$ INT

Multiset Problem

$x_i = 0, 1, 2, \ldots, k$

$$(x^0 + x^1 + x^2 + \cdots)^n = g(x) = \binom{n-1}{0}x^0 + \cdots + \binom{n+k-1}{k}x^k + \cdots$$

$$\binom{a}{b} = \binom{a}{a-b} \Longleftrightarrow \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

Example 2

$$x_1+2x_2+3x_3+4x_4=k \quad (\star)$$

$$x_i \geq 0 \quad \text{INT}$$

$$y_1 = 1\cdot x_1$$

$$\begin{cases} y_1 = x_1 \\ y_2 = 2x_2 \\ \vdots \\ y_4 = 4x_4 \end{cases}$$

$$(\star) \quad y_1+y_2+y_3+y_4=k$$

$$\left.\begin{array}{l} y_1 = 0, 1, \dots \\ y_2 = 0, 2, 4, 6, \dots \\ y_3 = 0, 3, 6, 9, \dots \\ y_4 = 0, 4, 8, 12, \dots \end{array}\right\} \quad g(x) = \underbrace{(1+x+x^2+\dots)}_{y_1}\underbrace{(1+x^2+x^4+\dots)}_{y_2}\underbrace{(1+x^3+x^6+\dots)}_{y_3}\underbrace{(1+x^4+x^8+\dots)}_{y_4}$$

Example 3) The distribution of 10 identical balls into 3 distinguished classes with even number of balls in each cell

$x_i = \#$ balls in cell $i = 1, 2, 3$

$x_1 + x_2 + x_3 = 10 = k$

$x_i = 0, 2, 4, 6, \ldots$ $\Big\} \rightarrow g(x) = (x^0 + x^2 + x^4 + x^6 + \ldots)^3$

Example 4) HW ...

A roll of 3 distinct dice yields a sum of 11

$x_1 + x_2 + x_3 = 11$

$x_1 = 1, 2, 3, 4, 5, 6$ $\Big\}$ $g(x) = (x + x^2 + \ldots + x^6)^3$

Example 5

$x_1 + x_2 + \cdots + x_n \leq k$ $x_1 + x_2 + \cdots + x_n + s = k$

$x_i \geq 0$ INT add slack variable $s, x_i \geq 0$ INT

$g(x) = (1 + x + \cdots)^{n+1}$

$x_1 + x_2 + \cdots + x_n < k$

$x_1 + x_2 + \cdots + x_n + s = k$

$x_i \geq 0$ INT

$s = 1, 2, 3, \ldots$

$g(x) = (1 + x + \cdots)^n (x + x^2 + \cdots)$

x^0 is missing since $s > 0$

Example 6

ways of choosing 5 distinct numbers from the set $\{1, \ldots, n\}$

$$1 \le n_1 < n_2 < n_3 < n_4 < n_5 \le n$$

Let

$$x_1 = n_1$$
$$x_2 = n_2 - n_1$$
$$x_3 = n_3 - n_2$$
$$x_4 = n_4 - n_3$$
$$x_5 = n_5 - n_4$$
$$x_6 = n - n_5$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = n$$

$x_i = 1, 2, \ldots \quad i = 1, 2, \ldots, 5$

$x_6 = 0, 1, 2, \ldots$

$$g(x) = (x + x^2 + \ldots)^5 (1 + x + x^2 + \ldots)$$

Formula Sheet

$$(1+x+x^2+\cdots) = \frac{1}{1-x}$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^1$$

$$(1+x+\cdots)^n = \binom{n-1+0}{0} + \binom{n-1+1}{1}x + \cdots + \binom{n-1+k}{k}x^k + \cdots$$