with R. | W/O Rep.

k-seq

$$n^k$$

(A, A) (B, A) (C, A)
(A, B) (B, B) (C, B)
(A, C) (B, C) (C, C)

k-perm

$$\frac{n!}{(n-k)!} = P(n; k)$$

(A, B) (B, A)
(A, C) (C, A)
(B, C) (C, B)

(,) ARRANGE

k-multiset

$$\binom{n+k-1}{n-1}^{*}$$

{A, A} {A, B}
{A, C} {B, B}
{B, C} {C, C}

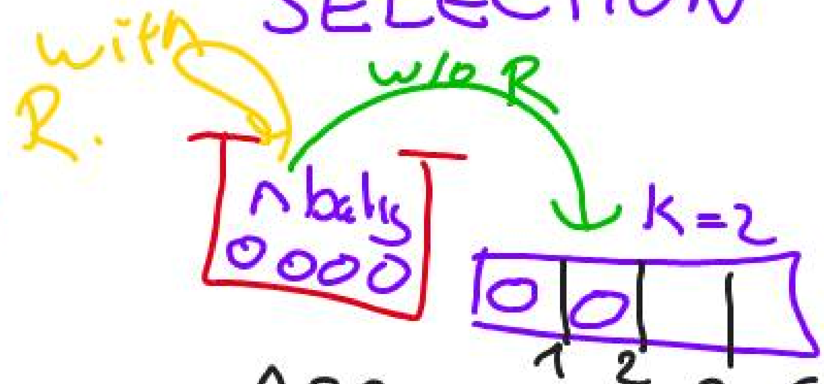
k-comb

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

{A, B} {B, C}
{C, A}

SELECT { , }

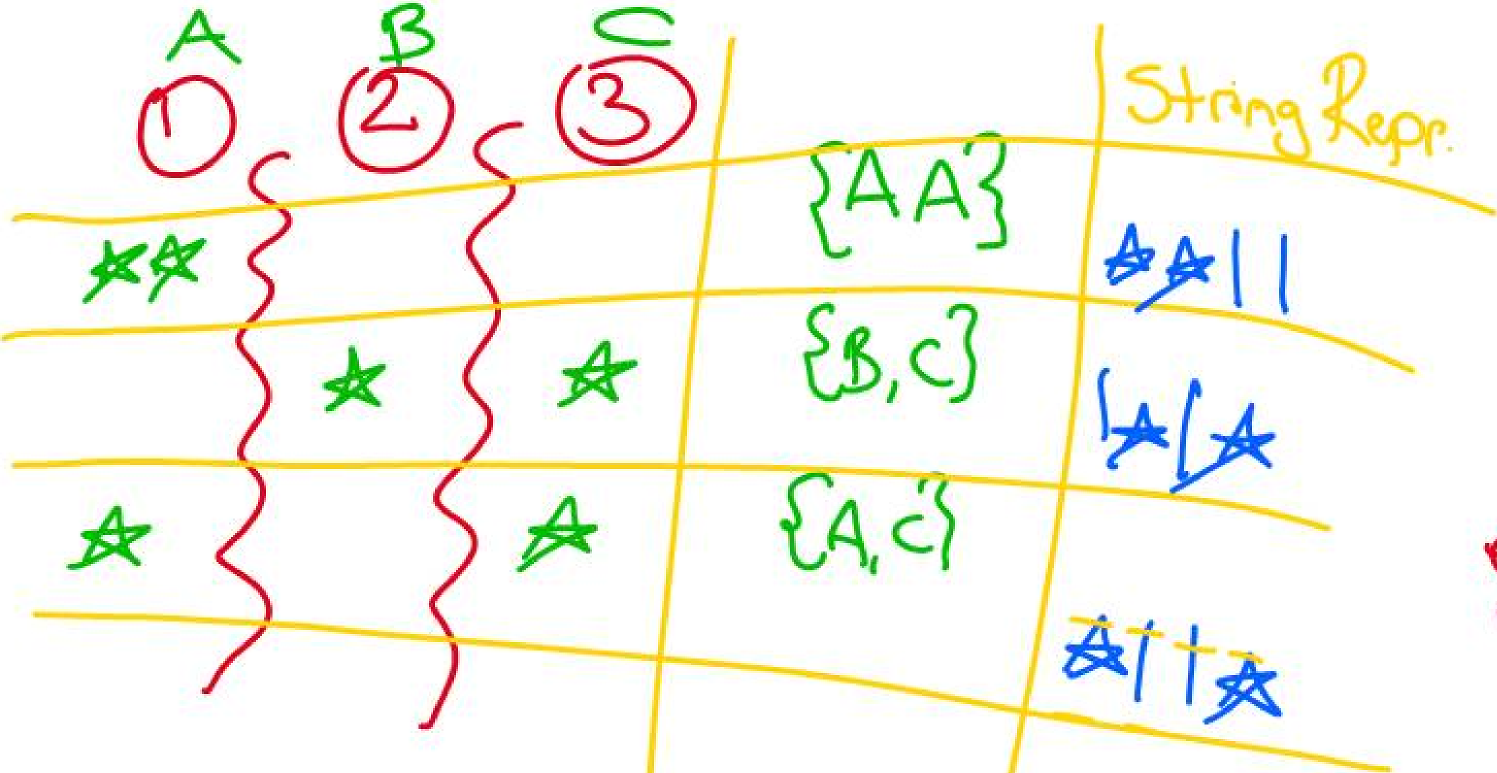
SELECTION



ARRANGEMENT

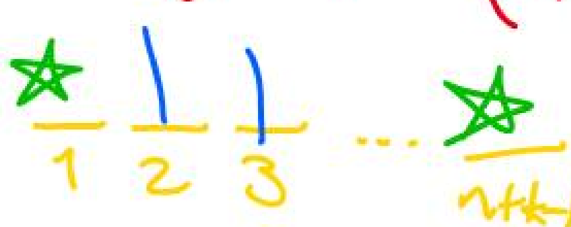
E.G. $n = 3$ $\{A, B, C\}$
$k = 2$

(*) ⤳ next page



I have $(n-1)$ many | "seperators"
+ k many ☆ "stars"
$n+k-1$ many places

place



Selecting $(n-1)$ sep. out of $n+k-1$ places

$$\binom{n+k-1}{n-1}$$

What is the probability of having a full house in a poker game

$\rightarrow$ 5 cards out of 52

$\rightarrow$ 3-of-a-kind
2-of-a-kind

AA KKK

$\exists \binom{52}{5}$ different hands

S1. Pick the triplet's face $\binom{13}{1} = 13$

S2. Choose 3 out of 4 cards of the same face $\binom{4}{3}$

S3. Pick the pair's face $\binom{12}{1}$

S4. Choose 2 out of 4 $\binom{4}{2}$

RULE OF PRODUCT

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

The Birthday Problem

What is the probability of having at least one pair of people with the same birthday? in a class of $24 = n$?

$$P(\text{one match}) = 1 - P(\text{no match})$$

$$P(\text{no match}) \stackrel{?}{=} \frac{\binom{365}{24}}{365^{24}}$$

EXERCISE

$$\#(\text{no match}) = 365(364)(363)\cdots(365-24+1) \stackrel{?}{=} \binom{365}{24} \stackrel{?}{=} P(365;24)$$

SAMPLE QUIZ 1: Open notes!

There are 28 regular and 11 irregular students who are currently taking IE 454. Out of these students, only 20 are female. The lecture is scheduled in room A201 on March, 2. Being in the state of rush, 5 students enter the class after 9:45.

1. How many different ways of having 5 students late for the quiz?

$$\binom{39}{5}$$

2. What is the probability of having 2 irregular students (and henceforth 3 regular) late?

$$\frac{\binom{11}{2}\binom{28}{3}}{\binom{39}{5}}$$

3. What is the probability of having at least 2 gentlemen being late?

$$1 - P(\text{no male}) - P(\text{one male})$$

$$= 1 - \frac{\binom{20}{5}\binom{19}{0}}{\binom{39}{5}} - \frac{\binom{20}{4}\binom{19}{1}}{\binom{39}{5}}$$

4. What is the probability that all will sit at the rear row? The classroom has five rows for seating.

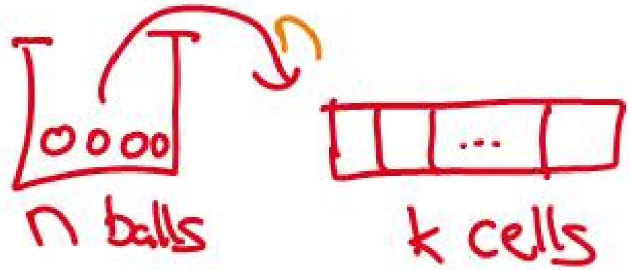
Multi-set

Students $k=5$

Rows $=5$ $= n$

$$\frac{1}{\binom{5+5-1}{5-1}}$$

OCCUPANCY PROBLEMS



CASE	Balls Distinguished	Cells Dist.	Can Cells be Empty	FORMULAE
1a	✓	✓	✓	k^n
1b	✓	✓	✗	$k!\, S(n;k)$
2a	✗	✓	✓	$\binom{k+n-1}{n}$
2b	✗	✓	✗	$\binom{n-1}{n-k}$
3a	✓	✗	✓	$\sum_{i=0}^{k} S(n;i)$
3b	✓	✗	✗	$S(n;k)$
4a	✗	✗	✓	★
4b	✗	✗	✗	★

Case 1a Balls $\longleftrightarrow$ D
Cells $\longleftrightarrow$ D
can be empty

$n=4$ a b c d

$k=3$; 1, 2, 3

balls

1	abcd	a	ab	
2		b	d	. . .
3		cd	c	

For each ball, we have $k=3$ alternatives

for n balls

Rule of product

$$\underset{\text{ball}}{\overset{1^{st}}{k}}\ \underset{\text{ball}}{\overset{2^{nd}}{(k)}} \cdots \overset{\text{last}}{(k)} = k^n$$

Case 2a Balls ← identical $n=4$ a, a, a, a

Cells ← Dist. $k = 1^{st}, 2^{nd}, 3^{rd}$

↳ can be empty

$$\binom{a}{b} = \binom{a}{a-b}$$

1	aaaa	aaa		a	
2		a	aa	aa	
3			aa	a	

Multi-set problem with $n=4$ STARS

$k-1$ many column seperators

Note that n & k here are switched!

$$\binom{k+n-1}{k-1} = \binom{k+n-1}{n}$$

Case 2b

Balls: identical $n=4$: a, a, a, a

Cells: disting. $k=3$ $1^{st}, 2^{nd}, 3^{rd}$

↳ CAN NOT be empty

$n \geq k$

1	~~a~~a	~~a~~	~~a~~
2	~~a~~	~~a~~ a	~~a~~
3	~~a~~	~~a~~	~~a~~a

We must put one ball to each cell.

What is left is $(n-k)$ balls to be distributed as Case 2a

$n' = n-k$

$$\binom{n'+k-1}{n'} = \binom{n-1}{n-k}$$

$$n'+k-1 = (n-\cancel{k})+\cancel{k}-1 = n-1$$

Stirling's Number of the Second Kind

$$S(n;k) = \frac{1}{k!} \underbrace{\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n}_{\alpha}$$

Case 3b

$n=4 \quad k=2$

$$\alpha = (-1)^0 \binom{2}{0} (2-0)^4 + (-1)^1 \binom{2}{1} (2-1)^4 + (-1)^2 \binom{2}{2} (2-2)^4$$

$0! = 1 \Rightarrow \alpha = +(1)2^4 - (2)1^4 + (1)0^4$

$\Rightarrow \alpha = 16 - 2 = 14$

$$S(4;2) = \frac{14}{2!} = 7 \longrightarrow$$

Case 1b Cells distinguished $k!\,S(n;k)$

adc	b
adb	c
~~abcd~~	cant!
dc	ab
a,b	d,c > same
a,c	d,b
a,b,c	d
a	d,b,c
ad	bc

Case 4

Let $n=7$

How many ways are there to partition integer 7 with sums

$7 = 7$ single

$= 6+1$

$= 5+2$

$= 4+3$ double

$= 5+1+1$

$= 4+2+1$ triple

$= 3+2+2$

$= 2+2+2+1$ 4-way

$= 2+2+1+1+1$

$\vdots$

$= 1+1+1+1+1+1+1$ 7-way

$\nexists$ close form formula

Must count one by one

($\star$)