Department of
Industrial Engineering

IE 454 Combinatorial Analysis

http://ie454.cankaya.edu.tr
Fall 2010 Tuesday 9:40-12:30 A201
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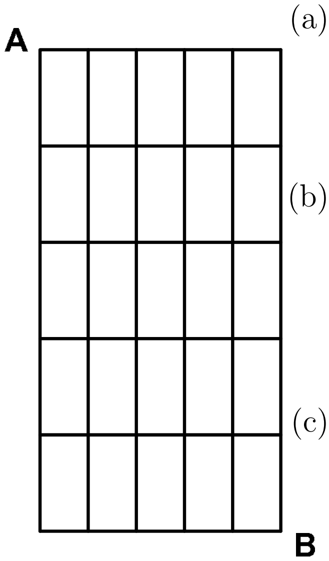
Solutions for HOMEWORK 2

Q1.



(a) *How many different ways are there to pick 4 different cells in this 5 by 5 mesh?*

$$\binom{25}{4}$$

(b) H*ow many different ways are there to pick 4 cells in sequence as* 1^{st},2^{nd},3^{rd}, *and* 4^{th} *such that one can pick the same cell in at most four times in this 5 by 5 mesh?*

$$25^4$$

(c) *How many different ways are there to pick 15 cells in this 5 by 5 mesh such that exactly 3 cells in each column are selected?*

$$\binom{5}{3}^5$$

(d) *How many different ways are there to pick 3 cells in this 5 by 5 mesh such that no two cells in the same row and column are selected?*

$$5(4)3\binom{5}{3} = 240$$

(e) *Consider that you are at point B. The side figure is the road-map of a district. You may use horizontal or vertical streets. How many different routes of length 10 from B to A are there?*

$$\binom{10}{5}$$

Q2. *Capacitated bins and indistinguishable balls:*

Consider n indistinguishable balls and m bins, where each bin has a capacity of $c(i)$ ($i = 1, \ldots, m$) balls. Note that $n \leq \sum_i c(i)$; one or more bins may have room for all n balls; we don't care which balls are in which bins, nor do we distinguish between positions in the bins; and bins need not be occupied.

Let $N(k)$ denotes the number of ways of packing k balls into m bins with capacities $c(i)$.

(a) *If we have 5 bins with capacities 3,2,5,4,2 respectively, then what will be* $N(1)$*?*

$$\binom{5}{1} = 5$$

(b) *If we have 5 bins with capacities 3,2,5,4,2 respectively, then what will be* $N(2)$*?*

$$\binom{2+5-1}{2} = \binom{6}{2} = 15$$

(c) *If we have 5 bins with capacities 3,2,5,4,2 respectively, then what will be* $N(14)$*?*

$$\binom{2+5-1}{2} = \binom{6}{2} = 15$$

(d) *Prove* $N(k) = N(c-k)$ *combinatorially, where c is the total capacity* $c(1)+c(2)+\cdots+c(m)$*:*

Naturally, we have $N(k) = N(c-k)$, because the distribution of empty spaces is symmetrical with the distribution of balls.

(e) *If we have 5 bins with capacities 3,2,5,4,2 respectively, then what will be* $\sum_{k=0}^{16} N(k)$*?*

$N(0) = 1 = N(16)$, $N(1) = 5 = N(15)$, $N(2) = 15 = N(14)$, but the rest is rather hard to compute. There is another way of looking things!

For each bin, $\begin{cases} \text{no ball at all} \\ \text{one ball} \\ \vdots \\ c(i) \text{ balls} \end{cases}$ there are $c(i) + 1$ possibilities. Then,

$$\sum_{k=0}^{16} N(k) = 4(3)6(5)3 = 1080$$

(f) *What is* $\sum_{k=0}^{c} N(k)$ *in general, where c is the total capacity?*

$$\sum_{k=0}^{c} N(k) = [c(1)+1][c(2)+1]\cdots[c(m)+1],$$

where c is the total capacity $c(1) + c(2) + \cdots + c(m)$. The total of these values of $N(k)$ is 1080, which can be computed directly as a function of the individual bin capacities as $(3+1)(2+1)(5+1)(4+1)(2+1) = 1080$.

(g) In how many ways can the n balls be distributed in the m bins? [BONUS]

$$n(k) = \sum_{t=0}^{m} (-1)^m \sum_{t=0}^{\binom{m}{t}} \binom{m+k-s(t,j)-1}{m-1},$$

where $s(t,j)$ is the j^{th} sum of t "capacity+1"s.

Q3.

$$x_1 + x_2 + x_3 + x_4 + x_5 = n$$

$$x_1, x_2, x_3, x_4 = 1, 2, 3, 4, 5, 6;\ 0 \le x_5 < 6 \text{(Strict inequality!)}.$$

(a) *Find bounds (lower, upper) on n.*

$$4(1) + 0 = 4 \le n \le 29 = 4(6) + 5$$

(b) *Construct the generating function.*

$$g(x) = (x + x^2 + \cdots + x^6)^4 (1 + x + \cdots + x^5) = x^4 (1 + x + \cdots + x^5)^5 = x^4 k(x),$$

where $k(x) = (1 + x + \cdots + x^5)^5$.

(c) *Find the probabilities for n=5 and for n=11.*

$k(x) = (1 + x + \cdots + x^5)^5 = [1 - x^6]^5 [1 + x + x^2 + \cdots]^5 =$ <BINOMIAL> <MULTISET>

$$k(x) = \left[\binom{5}{0} - \binom{5}{1}x^6 + \binom{5}{2}x^{12} - \binom{5}{3}x^{18} + \binom{5}{4}x^{24} - \binom{5}{5}x^{30}\right] \left[\binom{4+0}{0} + \binom{4+1}{1}x + \cdots + \binom{4+k}{k} + \cdots\right].$$

$n = 5$: We are interested in $n = 1$ in $k(x)$.

$$\binom{5}{0}\binom{5}{1} = 5.$$

$n = 11$: We are interested in $n = 7$ in $k(x)$.

$$-\binom{5}{1}\binom{5}{1} + \binom{5}{0}\binom{4+7}{7} = 330 - 25 = 305.$$

Thus, the probabilities we are after are $\frac{5}{6^5}$ and $\frac{305}{6^5}$, respectively.

Q4. *Fibonacci numbers:*
Find an explicit formula for Fibonacci numbers:

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2},\ n = 2, 3, \cdots$$

Characteristic equation: $c^2 - c - 1 = 0 \Rightarrow c_1 = \frac{1+\sqrt{5}}{2},\ c_2 = \frac{1-\sqrt{5}}{2}$. Since we have two distinct roots, the general solution is

$$F_n = Ac_1^n + Bc_2^n = A\left[\frac{1+\sqrt{5}}{2}\right]^n + B\left[\frac{1-\sqrt{5}}{2}\right]^n.$$

$$F_0 = 0 \Rightarrow A + B = 0 \ (1)$$
$$F_1 = 1 \Rightarrow (1+\sqrt{5})A + (1-\sqrt{5})B = 2 \ (2)$$

Thus,

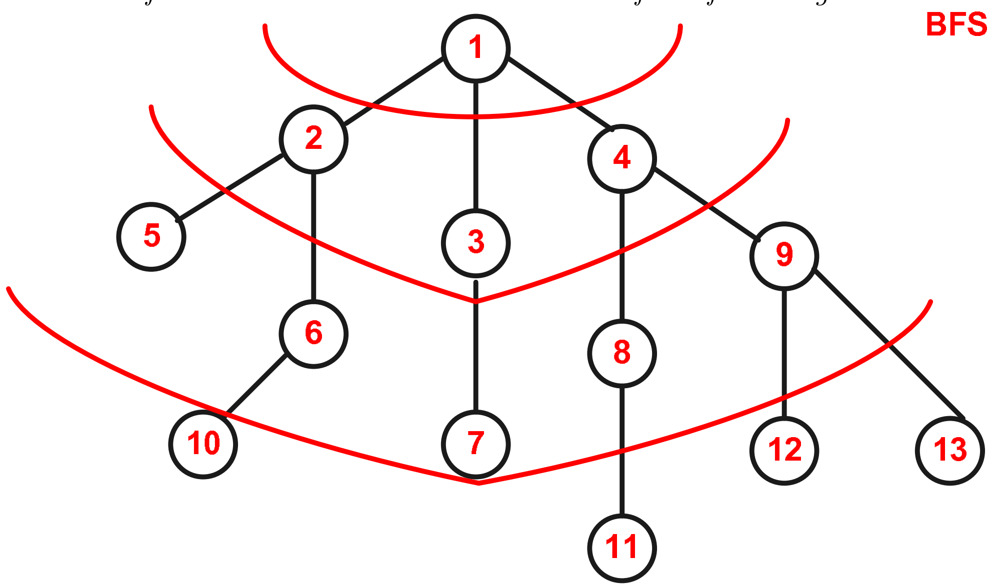
$$A = \frac{1}{\sqrt{5}}, B = -\frac{1}{\sqrt{5}}$$

yielding

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n.$$

Q5. *Tree traversal:*

(a) *Find the bfs numbers and write in the nodes of the following tree:*



(b) *Find the dfs numbers and write in the nodes of the following tree:*